

EM-D Stream: Mathematical Supplement

Derivations for the field topology, momentum transfer, radiation bound, circuit condition, magnetic-sail standoff and hull loading, with the assumptions each one rests on.

Sophia and Thomas Garren

SOPHIA XT LLC

18 August 2026, reissued 19 September 2026 · EMD-MATH-001 Revision D · Design study, pre-build

sophiaxt.com

CC BY 4.0. Code referenced herein is Apache 2.0.

Abstract

This supplement derives every equation the EM-D design study runs on, from first principles and in full: the field topology of a dynamo ring and the null condition that sets its geometry, the momentum a travelling field transfers to a proton, the radiation bound on a reflector-fed aperture, the circuit condition that makes the drive realisable, magnetic-sail standoff against the solar wind, and hull loading under Maxwell stress. Twenty numbered results, M1 to M20, are referenced by label throughout the document set, so any quantity in the specification can be traced to the algebra that produced it and to the assumption it rests on. Section 3 gathers those assumptions and states the conditions under which each stops holding, which makes the derivations checkable rather than merely quotable. Provenance is explicit throughout: every numerical value here is either the output of a named simulation or a laboratory measurement made by another group and cited as such. No EM-D vehicle or bench article has been built. This is Revision D, companion to the Revision D specification. It supersedes the 18 August 2026 issue, which was companion to Revision C, carried a Mars-transit column computed at the Revision C dry mass of 37.4 t, and closed with a render of the Revision C vehicle including a 100 kWe reactor, a shadow shield and a 2.2 t superconducting coil. None of that hardware exists in Revision D, whose configuration is 28.9 t with a 20 kg rotating-magnetic-field antenna and no reactor.

1. Revision status and companion documents

This is Revision D of EMD-MATH-001. It supersedes the issue of 18 August 2026 in its entirety. That issue was a companion to EMD-SPEC-001 Revision C and inherited two Revision C figures: a transit table computed at 37.4 t dry mass, and a vehicle render carrying a reactor, a shadow shield and a superconducting coil. Both are marked in this reissue, the transit column withdrawn and the render moved to an appendix and labelled. The derivations themselves are unchanged, because the configuration change from Revision C to Revision D is a change of hardware and not of physics: the magnetic moment is now carried by ambient plasma driven from an antenna rather than by a coil aboard, and the equations for what a moment does are the same either way.

The design of record is EM-D Stream: Design Specification (EMD-SPEC-001 Revision D). The bench experiment that would test the central claim, with its error budget and acceptance criteria, is EM-D Stream: Test Plan and Validation Record (EMD-TEST-001 Revision D). All three documents carry revision letter D. Where this supplement quotes a numerical result it names the simulation that produced it, by name only.

2. Notation and constants

Notation. Equations are labelled M1 to M20 and referenced by those labels throughout the document set.

Symbol	Quantity	Value or units
μ_0	vacuum permeability	$4\pi \times 10^{-7}$ H/m
c	speed of light	2.99792458×10^8 m/s
m_p	proton mass	$1.67262192 \times 10^{-27}$ kg
m	magnetic dipole moment	$A \cdot m^2$
R	dynamo ring radius	m
N	number of dipoles on the ring	dimensionless
n, v	solar wind density and speed at 1 AU	5×10^6 m ⁻³ , 400 km/s
ρ	solar wind mass density	$n m_p$, kg/m ³
η	radiated momentum fraction	dimensionless, ≤ 1
r_{mp}	magnetopause standoff radius	m
C_d	sail drag coefficient	dimensionless, see section 3

3. Assumptions and where the model stops being valid

Each derivation below is exact within its own assumptions, and this section states them and the domain over which each holds. It comes before the mathematics because that is where it is useful: a reader can fix the conditions once and then read every equation knowing what it is entitled to claim. It also makes the results falsifiable in the only way algebra can be. An assumption named with its breaking point can be attacked directly, and sections 12 and 13 say what the study loses if one of them gives way.

3.1 Point dipoles

Equation M1 is the field of a point dipole, and the ring sources are treated as point dipoles throughout. This is accurate where the field point is far compared with the coil radius. The flight geometry puts coils of order 0.05 m radius on a 10 m ring, where the approximation is comfortable. The bench article specified in the Test Plan section 2 puts coils of about 0.05 m radius on a 0.15 m ring, where the separation is only about three coil radii and the point-dipole form is an approximation rather than a limit. The resulting error in predicted null position and core field is not quantified anywhere in this record, and a bench result within a factor of three of prediction should not be read as a discrepancy in the physics before it has been.

3.2 Vacuum superposition and test particles

The summed field is the vacuum sum of the sources. Plasma currents are not fed back into the field. The momentum-transfer result of section 5 pushes 2 500 test protons through a prescribed field with a Boris integrator; it does not solve for the field the plasma itself produces. This holds in the low-plasma-pressure limit, where the induced current does not materially modify the applied field, and it is the single largest modelling assumption in the study. The self-consistent version was attempted with a Darwin particle-in-cell code and did not reach steady state: the ratio of electron cyclotron frequency to drive frequency is 186, which makes the problem stiff, and the run covered 0.29 of one drive period. That attempt is recorded in the Specification section 16.

3.3 Quasi-static treatment of the hull

Section 9 integrates the Maxwell stress tensor over a closed surface and uses the vanishing divergence of that tensor in a source-free static region. This requires the drive period to be long compared with light transit across the hull, which a kilohertz drive on a metre-scale hull satisfies by a wide margin. It does not extend to the radiating problem of section 6, which is treated in the far field instead, and it says nothing about the transient traction during a single sub-microsecond switching edge.

3.4 Far field, uniform illumination, real excitation

The aperture results evaluate M10a in the far field for a uniformly illuminated aperture with real excitation amplitudes. Feed blockage, illumination taper, surface error, polarisation and near-field behaviour are not modelled. The

two-hemisphere split of M11 also ignores rim spillover, which is why the computed values fall slightly below it and why the residual at 40 dB front-to-back is spillover-limited rather than backlobe-limited.

3.5 The sail inflow

Equations M16 to M18 balance the vehicle's dipole field pressure against the solar wind ram pressure and treat the momentum flux intercepted by the standoff disc as thrust through a drag coefficient. Three things follow. First, the tabulated thrusts correspond to $C_d = 1$, which is complete momentum capture over the standoff disc and is an upper bound rather than an expected value; the source material does not state the coefficient it used. Second, pressure balance sets plasma pressure equal to magnetic pressure at r_{mp} by construction, so the vacuum dipole expression for $B(r)$ is being evaluated exactly where a vacuum dipole is least defensible. Third, the wind is treated as cold, collisionless and unmagnetised, so the interplanetary field enters nowhere in the thrust; the earlier revision's error of using interplanetary magnetic pressure instead of ram pressure, a factor of 134, is described in section 8.

3.6 Distance scaling of sail thrust

Mission figures elsewhere in the set apply a falloff of $r^{-4/3}$ with heliocentric distance. Slough states instead that the bubble expands as the wind thins, giving constant force with distance. At 30 AU the two laws differ by a factor of 96. Nothing in this supplement verifies either. Any transit time beyond Mars inherits that unresolved choice.

3.7 Statistics of the particle ensemble

Significances quoted in section 5 are against the ensemble noise floor, taken as the random-walk value, the square root of N times the per-particle momentum scale. Using N times that scale would understate the floor by the square root of N , which is 50 for this ensemble, and would convert noise into apparent signal. These are numerical significances. They say the result is not an artefact of ensemble size and say nothing about whether the model describes a real plasma.

3.8 Structural treatment

Section 9.1 uses thin-wall hoop stress in isotropic aluminium against a 270 MPa yield with no factor of safety and no allowable below yield. Fatigue under kilohertz cycling, structural modes, joints, penetrations and thermal stress are not analysed. The ceiling it produces is therefore an upper bound on moment, not a design allowable.

3.9 Where the model stops

- At the laboratory-to-flight density gap. Every sail number rests on a measurement taken against a 50 km/s wind source at high density; flight is 400 km/s at roughly a millionth of the density. Nothing here bridges six orders of magnitude in density.
- Above the low-plasma-pressure limit, where the prescribed-field treatment of section 5 fails and a self-consistent solution is required. That solution has not been obtained.

- Outside the far field of the aperture, and outside the quasi-static limit for the hull.
- At any question of stability. No instability analysis of the inflated bubble, of the field-reversed core or of the exhaust plasmoid appears anywhere in this study.
- At loss mechanisms. Charge exchange, radiative losses, wall recombination in the bench article and differential

charging of the flight vehicle are not modelled.

4. Field-reversed core: the null condition

The central topological result. N identical dipoles of moment m along z sit on a ring of radius R in the plane $z = 0$.

4.1 Step 1, the dipole field

$$(M1) \quad B(r) = (\mu_0 / 4\pi) * [3\hat{r}(\hat{m} \cdot \hat{r}) - \hat{m}] / r^3$$

4.2 Step 2, geometry on the axis

For a field point at $(0, 0, z)$ and a source at radius R in the plane, the separation vector is $(-R \cos(\phi), -R \sin(\phi), z)$, so:

$$(M2a) \quad r = \sqrt{R^2 + z^2}$$

$$(M2b) \quad \hat{m} \cdot \hat{r} = m \cdot z / r$$

$$(M3) \quad B_z(z) = N * (\mu_0 m / 4\pi) * (3z^2 - r^2) / r^5$$

4.3 Step 3, transverse cancellation

Summing over ϕ uniformly distributed, the x and y components cancel by symmetry for $N \geq 2$. Only the axial component survives:

4.4 Step 4, substitute the separation

$$(M4) \quad B_z(z) = N * (\mu_0 m / 4\pi) * (2z^2 - R^2) / (R^2 + z^2)^{5/2}$$

4.5 Step 5, the null

The denominator is strictly positive, so the numerator alone sets the null:

$$(M5) \quad B_z = 0 \quad \Leftrightarrow \quad 2z^2 = R^2 \quad \Rightarrow \quad z_{\text{null}} = \pm R / \sqrt{2}$$

Consequence. Equation M5 contains neither m nor N . Current sets field strength; geometry sets field topology. No increase in drive current moves the nulls outward. In vacuum the topological scale of the machine equals its hardware scale, so a field structure larger than the hardware requires an external current carrier.

Verification. For $R = 10$ m, M5 predicts ± 7.0711 m. Numerical root-finding on a 1 201-point axial sample of the summed field locates sign changes at ± 7.0711 m. Between the nulls the axial field is negative: the core field is reversed

relative to the moments producing it, which is a field-reversed configuration. Computed: `emd_rubberband.py`. This is agreement between an analytic expression and a numerical evaluation of the same expression, which verifies the code and not the physics.

5. Why momentum transfer requires a translating pattern

Induction couples to the plasma through the electric field, which follows from the vector potential of the sources:

$$(M6a) \quad A(r, t) = \sum_i (\mu_0 / 4\pi) * m_i(t) \times (r - r_i) / |r - r_i|^3$$

$$(M6b) \quad E = -dA/dt$$

Consider uniform amplitude modulation, in which every moment shares one time profile $f(t)$. Since M6 is linear in

each moment:

$$(M7) \quad m_i(t) = f(t) * m_i(0) \quad \Rightarrow \quad E = -(df/dt) * A_{\text{unit}}(r)$$

The vector potential of an axial dipole is purely azimuthal, because the cross product of an axial moment with the position vector points along the azimuthal direction. Therefore the induced electric field is azimuthal everywhere, and the force per unit charge has no axial component at any point.

this, because the asymmetry resides in B while the momentum transfer is performed by E . Axial thrust requires the field pattern to translate, which means sources fired in temporal sequence, giving each source an independent phase.

Result. A breathing mode drives azimuthal current and imparts angular momentum only. Geometric asymmetry cannot repair

Computed. Boris-pusher integration including both E and B , 2 500 test protons, time step 5×10^{-8} s, in a prescribed field. These are simulation outputs and not measurements.

Test-particle ensemble results. Computed: emd_vectoring.py. The noise floor is the random-walk value; see section 3.7.

Drive ensemble	Axial momentum change, kg·m/s	Numerical standing
Symmetric control, 5 seeds	$+4.27 \times 10^{-21}$	0.47σ , consistent with null
Zero delay, breathing	$+1.33 \times 10^{-20}$	2.0σ , null
Wave toward $-z$	$+5.42 \times 10^{-20}$	8.1σ
Wave toward $+z$	-4.71×10^{-20}	7.0σ

the momentum fraction over the far-field intensity:

6. The radiation bound

For any radiating system the momentum carried by the electromagnetic field per unit energy is exactly $1/c$. Defining

$$(M8a) \quad \eta = \left| \int I(\hat{n}) \hat{n} d\Omega \right| / \int I(\hat{n}) d\Omega$$

$$(M8b) \quad F = \eta * P_{\text{rad}} / c$$

$$(M9a) \quad \eta \leq 1 \text{ by the triangle inequality, hence}$$

$$(M9b) \quad F \leq P/c = 3.336 \times 10^{-6} \text{ N per kW radiated}$$

6.1 Why a bare aperture gives exactly zero

A uniformly illuminated elliptical aperture with semi-axes a and b has far field:

$$(M10a) \quad E(\theta, \phi) \sim 2J_1(u)/u$$

$$(M10b) \quad u = k * \sqrt{(a \sin(\theta) \cos(\phi))^2 + (b \sin(\theta) \sin(\phi))^2}$$

Since u depends on the polar angle only through its sine, the pattern satisfies $I(n) = I(-n)$. The integrand in M8a is then odd and the integral vanishes identically, giving $\eta = 0$. This is exact and is not a small number that better engineering could

raise. It is why the reflector is the mechanism rather than an improvement to it.

6.2 The reflector

Suppressing the rear hemisphere by power ratio r and splitting M8a into hemispheres of equal forward and backward flux:

$$(M11) \quad \eta = (I_0 - r I_0) / (I_0 + r I_0) = (1 - r) / (1 + r)$$

The closed form against the computed aperture integral. Computed values fall slightly below M11 because a pure two-hemisphere split does not capture rim spillover; the residual at 40 dB, 0.97656, is spillover-limited rather than backlobe-limited. Computed: emd_aperture.py.

Front-to-back ratio	r	η from M11	η computed
0 dB		1.000	0.000
10 dB		0.100	0.818
20 dB		0.010	0.980
30 dB		0.001	0.998

6.3 Conversion to electrical power

$$(M12) \quad F = \eta_{\text{beam}} * \eta_{\text{conv}} * P_{\text{elec}} / c$$

With a beam fraction of 0.9748 and a magnetron at 0.85 conversion efficiency, the delivered figure is 2.764×10^{-6} N per

kW electrical. Quoting thrust per kilowatt radiated rather than per kilowatt electrical was correction 7 of the register in the Specification section 17, and was 15 to 30% optimistic.

7. The circuit condition

Force on a current-carrying conductor in an external field:

$$(M13) \quad F = I * \int dl \times B$$

$$(M14) \quad \text{uniform } B \text{ gives } F = I * \left(\int dl \right) \times B = I * 0 \times B = 0$$

$$(M15) \quad \text{the torque does not vanish: } \tau = m \times B, \text{ with } m = I * A$$

The closed-path integral of the displacement element vanishes identically, which is why M14 is zero and not merely small.

Result. A closed loop in a uniform field receives a torque and no net force: it aligns like a compass needle without translating. Net force requires either a field gradient, which at the interplanetary

gradient of 3.34×10^{-20} T/m yields about 10^{-10} N for a 100 m coil at 100 kA and is useless, or an open conductor whose circuit closes externally. In space the only available external conductor is ambient plasma. That is the entire plasma requirement: a current needs a return path, and vacuum has infinite resistance.

Verification. Direct integration of M13 over 20 000 segments at loop radii of 1 to 20 m and currents of 1 to 100 kA returns

```
(M16a) B(r_mp)^2 / (2*mu0) = rho*v^2
(M16b) B(r) = mu0*m / (4*pi*r^3)

(M17) r_mp = [ mu0*m^2 / (32*pi^2*rho*v^2) ]^(1/6)

(M18) F = Cd * rho*v^2 * pi*r_mp^2
```

Correction, a factor of 134. An earlier revision computed M18 using the interplanetary field's magnetic pressure, 9.947×10^{-12} Pa, rather than the wind's ram pressure, 1.338×10^{-9} Pa. Equation M16a makes the correct choice explicit: at the standoff radius

net force at machine precision, 10^{-21} to 10^{-18} N, in every case. Computed: emd_hull.py.

8. Magnetic sail: standoff and thrust

The standoff radius is defined by pressure balance between the vehicle field and the solar wind ram pressure:

the two pressures are equal by definition of r_{mp} , and the momentum delivered is the ram flux. The ratio is 134.5, and every thrust and transit figure preceding Revision C was understated by that factor.

Standoff and thrust at 1 AU. Moment scales as the cube of the bubble radius and thrust as its square, both from M17 and M18. Computed: emd_corrected.py. The superseded issue of this supplement carried a fourth column of Mars transit times computed at 37.4 t, which is the Revision C dry mass; that column is withdrawn. Transit times at the Revision D dry mass of 28.9 t are in the Specification section 11 and have not been recomputed here.

Bubble radius	Moment required, $A \cdot m^2$	Thrust from M18
30 km	1.4×10^{13}	3.78 N
60 km	1.1×10^{14}	15.1 N
120 km	1.00×10^{15}	60.5 N
300 km	1.57×10^{16}	378 N

9. Hull loading

The Maxwell stress tensor for a magnetic field:

```
(M19) T_ij = (1/mu0) * ( B_i*B_j - 0.5*delta_ij*B^2 )
```

The traction on a surface element with outward normal n is the tensor contracted with that normal, carrying tension along the field and equal pressure across it. Integrating over a closed

surface enclosing no sources gives zero by the divergence theorem, since the divergence of the tensor vanishes in a source-free static region.

Grid convergence of the residual on the hull sphere, four dynamos at 40 degrees cant. Exactly second-order convergence to zero, which identifies the residual as quadrature error rather than force. Computed: emd_angles.py.

Grid, polar samples	Residual force, N	Ratio to previous
60	1.1573×10^{-2}	no previous grid
120	2.8889×10^{-3}	4.01
240	7.2194×10^{-4}	4.00
480	1.8047×10^{-4}	4.00

Forward and aft hemispheres carry equal and opposite axial traction to five significant figures at every cant angle: the hull is stretched, not propelled.

9.1 Structural limit

Peak traction scales as the square of the moment, and thin-wall hoop stress is $pR/2$:

```
(M20) sigma(m) = sigma_0 * (m / m_0)^2
```

Practical ceiling for a 2.2 m aluminium hull is approximately $3 \times 10^7 \text{ A} \cdot \text{m}^2$ per dynamo, quoted against yield with no factor of safety. Computed: emd_angles.py.

Moment per dynamo	Hoop stress	Fraction of aluminium yield
$1 \times 10^6 \text{ A} \cdot \text{m}^2$	0.04 MPa	0.01%
$1 \times 10^7 \text{ A} \cdot \text{m}^2$	3.69 MPa	1.4%
$1 \times 10^8 \text{ A} \cdot \text{m}^2$	369 MPa	137%

10. Appendix: superseded configuration render

The previous issue closed with a section headed Vehicle, carrying the render reproduced below and the description: forward, a 15 m superconducting coil on eight spars, then the four reflector-fed dynamos on a 3 m ring at 20 degrees cant; centre, a 4.4 m by 6 m habitat with service section; aft, 24 m²

of radiators, a 12 m boom, a shadow shield and a 100 kWe reactor. That is the Revision C vehicle. None of the coil, the shield or the reactor exists in Revision D. The render is kept because it is the only vehicle artwork in the document set and a reader will meet it elsewhere; it is kept here, in an appendix, rather than in the body, because a derivation document should not appear to endorse a configuration that has been withdrawn.

Superseded Revision C vehicle. This configuration has been withdrawn.

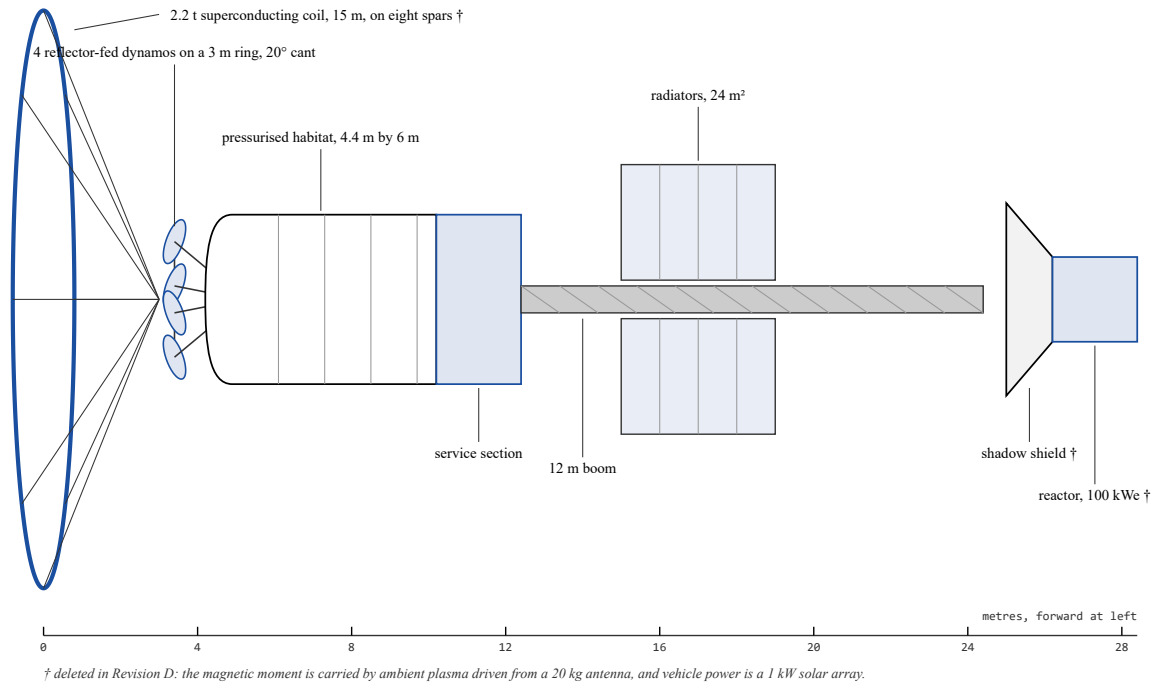


Figure M1. **Superseded Revision C vehicle. This configuration has been withdrawn.** The large ring forward is the 2.2 t, 15 m superconducting coil on its spars, with the four reflector-fed dynamos on the smaller ring behind it. The pressurised habitat and service section follow, then the radiator panels on the boom, the conical shadow shield and the reactor at the aft end. Revision D deletes the coil, the shield and the reactor: the magnetic moment is carried by ambient plasma driven from a 20 kg antenna, and vehicle power is a 1 kW solar array. No Revision D general-arrangement drawing has been produced, so this is the only vehicle artwork that exists and it is wrong. The configuration of record is the table in the Specification section 11.

11. What these derivations establish

- That the on-axis null of an aligned dipole ring sits at $R/\sqrt{2}$ independently of current and of the number of sources, so current buys field strength and never field extent. This is exact, and it is the reason a large field structure requires ambient plasma rather than a bigger coil.
- That uniform amplitude modulation of an axial dipole array cannot produce axial thrust in any geometry, because the induced electric field is azimuthal everywhere. This follows from the form of the vector potential and does not depend on the plasma model.
- That a bare aperture has a radiated momentum fraction of exactly zero by pattern symmetry, and that the closed form for a reflector, $(1 - r)/(1 + r)$, bounds what a given front-to-back ratio can deliver.
- That a closed current loop in a uniform field receives torque and no force, so a drive of this class requires an external current path.
- That the field of a dynamo cluster stretches its hull rather than moving it, with the residual confirmed as quadrature

error by second-order grid convergence.

12. What these derivations do not establish

- That any of the computed thrusts would be measured. Every result here is analysis. The only laboratory measurement in the document set was made by another group on different hardware. The experiment that would settle the central claim is specified in EM-D Stream: Test Plan and Validation Record (EMD-TEST-001 Revision D) and has not been run.
- That the plasma responds as modelled. Section 5 pushes test particles through a prescribed field. A real plasma carries current, that current modifies the field, and that coupling is absent here. The attempt to include it did not converge.
- That the sail thrust law holds. M18 uses a drag coefficient the source material does not state, and the tabulated values correspond to complete momentum capture, which is an upper bound. The distance dependence is disputed by the source of the measurement by a factor of 96 at 30 AU.
- That the aperture results survive a real antenna. Feed blockage, taper, surface error and polarisation are all outside M10a.
- That the structural ceiling is a design limit. It is yield with no factor of safety and no fatigue analysis, so it is an upper bound on moment and not an allowable.

13. Honest scope and limitations

This supplement is a derivation record for a design study of a vehicle that does not exist. Its value is that each number in the specification can be traced to an equation and each equation to an assumption, so that a reader who disagrees can find the exact line to disagree with. Its limitation is that tracing a number to an assumption does not make the assumption true.

- **Verification is not validation.** Where this document says two routes agree to six significant figures, or that a

numerical root matches an analytic one to four decimal places, it is reporting that the code solves the equations it was given. Nothing here tests whether those equations describe the physical situation.

- **One derivation was not repeated for this revision.** The transit times withdrawn from section 8 were computed at the Revision C mass. Revision D transit times exist in the Specification section 11 but the derivation behind them has not been reproduced here, so this supplement currently derives thrust but not transit.
- **No uncertainty propagation.** No derivation in this document carries an error bar. Inputs such as wind density, drag coefficient and conversion efficiency are treated as exact, and the sensitivity of the mission figures to them is not characterised.
- **Single implementation.** Every computed check was produced by the same set of scripts on one machine. Independent reimplementations would be worth more than any of the internal agreement figures quoted here.

Derivations verified against: `emd_rubberband.py` for section 4, `emd_vectoring.py` for section 5, `emd_aperture.py` for section 6, `emd_hull.py` for section 7, `emd_corrected.py` for section 8, and `emd_angles.py` for section 9. Files are named without paths.

References

- Slough, J. (2007). Plasma Magnet. IEPC-2007-015, 30th International Electric Propulsion Conference. Source of the measured antenna parameters and of the constant-force claim discussed in section 3.6.
- EM-D Stream: Design Specification (EMD-SPEC-001 Revision D). The design of record, and the source of every configuration number.
- EM-D Stream: Test Plan and Validation Record (EMD-TEST-001 Revision D). The experiment that would test the section 5 result, with error budget and acceptance criteria.
- EMD-MATH-001, issue of 18 August 2026, companion to EMD-SPEC-001 Revision C. Superseded in its entirety by this document.